Stability criteria for delayed neural networks

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(Received 3 June 2000; revised manuscript received 11 May 2001; published 12 October 2001)

In this paper, delay-independent global asymptotic and exponential stability for a class of delayed neural networks (DNN's) is investigated, and some criteria are established to ensure stability of DNN's by applying the Lyapunov direct method. These criteria are expressed by imposing constraints on weight matrices of the networks, and they are easy to verify and so are applicable in the design of DNN's. Comparisons between our criteria and some earlier results are also made; it is shown that our results generalize some existing criteria in the literature.

DOI: 10.1103/PhysRevE.64.051901 PACS number(s): 87.10.+e, 85.40.Ls, 05.45.−a, 43.80.+p

I. INTRODUCTION

Recently, the stability properties of delayed neural networks (DNN's) have attracted increasing attention. Among many existing research works, some deal with delayed Hopfield neural networks (DHNN's) with smooth activation functions [1–6], while others involve delayed cellular neural networks (DCNN's) with piecewise linear activation functions [7–13]. Some works consider stability independent delays [1–4,6–8,10–13], and others delay-dependent stability [5,9]. In this paper, we focus on the delay-independent global asymptotic or exponential stability of a class of delayed neural networks. Our delayed neural network model includes some well known models such as delayed Hopfield neural networks and delayed cellular neural networks [14] as special cases. In the design of a DNN, stability is a main concern; in many applications it is desired to design a DNN with unique equilibrium that is globally asymptotically stable (GAS) or globally exponentially stable (GES), by GAS (GES) we mean that any solution with any initial condition will converge (with an exponential speed) to unique equilibrium. In this paper we will derive some conditions for global asymptotic or exponential stability of DNN's by the Lyapunov direct method. This is achieved mainly by constructing Lyapunov functionals. Comparisons with some earlier results are also made, it is shown that our results generalize some existing conditions in the literature.

Consider the DNN model governed by a set of differential equations with delays,

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij}^0 f_j(x_j(t)) + \sum_{j=1}^{n} a_{ij}^{\tau} f_j(x_j(t-\tau_j)) + u_i, \quad i=1,2,\ldots,n, \quad (1)$$

or rewritten as

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + A^{\tau} f(x(t-\tau)) + u,$$

where $x(t)=(x_1(t),\ldots,x_n(t))^T$ is the state vector of the neural network, $C=\mathrm{diag}(c_1,\ldots,c_n)$ is a diagonal matrix with positive diagonal entries, $c_i>0$, $A=(a_{ij}^0)_{n\times n}$ is the weight matrix, $A^{\tau}=(a_{ij}^{\tau})_{n\times n}$ is the delayed weight matrix, $u=(u_1,\ldots,u_n)^T$ is the constant input vector, $\tau_j \geqslant 0$ are delays, and $f(x(t))=[f_1(x_1(t)),\ldots,f_n(x_n(t))]^T$, $f(x(t-\tau)) =[f_1(x_1(t-\tau_1)),\ldots,f_n(x_n(t-\tau_n))]^T$. For a conventional DCNN [14], all functions f_j's are the same and take the form of piecewise linear function as $f_j(x)=\frac{1}{2}(|x+1|-|x-1|)$. In Hopfield neural networks, f_j's are usually sigmoidal. In this paper, f_j's are allowed to be different and more general, satisfying the following conditions.

(A1) All $f_j(x)$'s are bounded and monotonic nondecreasing on *R*, the set of real numbers.

(A2) All $f_j(x)$'s satisfy the global Lipshitz condition, that is, there exist real numbers $k_j>0$, $j=1,2,\ldots,n$, such that $|f_j(x_1)-f_j(x_2)| \leqslant k_j|x_1-x_2|$ for arbitrary x_1, x_2.

Conventional DCNN's and DHNN's obviously satisfy these conditions.

Throughout this paper, we will use the following notations.

(i) $\sigma(z_1,z_2): R\times R \to R$ is defined as

$$\sigma(z_1,z_2)=\begin{cases} 1 & \text{if } z_1>0 \text{ or } z_1=0 \text{ and } z_2>0 \\ 0 & \text{if } z_1=z_2=0 \\ -1 & \text{if } z_1<0 \text{ or } z_1=0 \text{ and } z_2<0. \end{cases} \quad (2)$$

(ii) $\tau \overset{\mathrm{def}}{=} \max_{1\leqslant j\leqslant n}\{\tau_j\}$.

(iii) $K=\mathrm{diag}(k_1,\ldots,k_n)$, where the diagonal matrix with diagonal entries k_i.

(iv) *I*, is the $n\times n$ identity matrix.

(v) $\lambda_{\max}(M)$, is the largest eigenvalue of a symmetric matrix *M*.

(vi) $\|M\|_2$, is the spectral norm of a matrix *M*, i.e., $\|M\|_2 =\{\lambda_{\max}(M^TM)\}^{1/2}$.

(vii) $\|M\|_m$, is the Euclidean norm of a matrix $M =(m_{ij})_{n\times n}$, i.e., $\|M\|_m=(\Sigma_{i,j}m_{ij}^2)^{1/2}$.

(viii) $\|M\|_1$, is the column norm of a matrix *M* defined by $\|M\|_1=\max_j\{\Sigma_i|m_{ij}|\}$.

(ix) $\|M\|_\infty$, is the row norm of a matrix *M* defined by $\|M\|_\infty=\max_i\{\Sigma_j|m_{ij}|\}$.

(x) $\mu_\infty(M)$, is the matrix measure defined by $\mu_\infty(M) =\max_i(m_{ii}+\Sigma_{j\neq i}|m_{ij}|)$.

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(xi) $\mu_1(M)$, is the matrix measure defined by $\mu_1(M) = \max_j(m_{jj} + \Sigma_{i\neq j}|m_{ij}|)$.

(xii) $\mu_2(M)$, is the matrix measure defined by $\mu_2(M) = \lambda_{\max}[(M+M^T)/2]$.

When the delayed weight matix $A^\tau = 0$, model (1) reduces to the conventional neural networks without delays:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij}^0 f_j(x_j(t)) + u_i, \quad i = 1,2,\ldots,n. \tag{3}$$

Stability properties of such kinds of networks without delays have been extensively studied by many researchers; a large number of criteria for absolute stability (meaning for any arbitrarily chosen functions f_j and external inputs u_i, the system has a unique equilibrium which is GAS) have been derived, most of them given in terms of matrix norms or matrix measures of the weight matrix *A*. For example, some representative results are listed below (when assuming all $c_i = 1$, and $k_i = 1$, and using notations of this paper, and in some cases with slightly different assumptions on f_j).

(i) $\|A\|_2 = \{\lambda_{\max}(A^T A)\}^{1/2} < 1$; see Ref. [15].

(ii) $\|A\|_\infty = \max_i\{\Sigma_j|a_{ij}^0|\} < 1$; see Ref. [16].

(iii) $\mu_\infty(A) = \max_i(a_{ii}^0 + \Sigma_{j\neq i}|a_{ij}^0|) < 1$; see Ref. [17].

(iv) $\mu_1(A) = \max_j(a_{jj}^0 + \Sigma_{i\neq j}|a_{ij}^0|) < 1$; see Ref. [18].

(v) $\max_i\{a_{ii}^0 + \frac{1}{2}\Sigma_{j\neq i}(|a_{ij}^0| + |a_{ji}^0|)\} < 1$; see Ref. [19].

(vi) $\mu_2(A) = \lambda_{\max}\{(A+A^T)/2\} < 1$; see Ref. [20].

For any pair of a matrix norm $\|\cdot\|$ and a matrix measure $\mu(\cdot)$ induced by the same vector norm, it always holds that $\mu(\cdot) \leq \|\cdot\|$, thus among the above conditions, (iii) is less conservative than (ii), (vi) is less conservative than (i) and (v), while (iii), (iv) and (vi) are independent of each other. In theorem 5 of Ref. [21], the author established a sufficient condition under the constraint that the f_j's are differentiable and with bounded derivatives, which states that (translated to our notations), if there exists a diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i > 0$ such that the matrix

$$\mathrm{diag}\left(\frac{\alpha_1 c_1}{k_1}, \ldots, \frac{\alpha_n c_n}{k_n}\right) + \frac{1}{2}(\alpha A + A^T \alpha)$$

is negative definite, then network (3) is globally asymptotically stable. This condition is further extended to a more general case in the theorem 4 of Ref. [22] where constraint on activation functions is relaxed. This is expressed in our notations as follows: If every $f_j(x)$ is a locally Lipschitz continuous mapping with $f_j(0) = 0$, and there exist constants $0 < k_j < \infty$, $j = 1,2,\ldots,n$, such that

$$0 \leq \frac{f_j(x_1) - f_j(x_2)}{x_1 - x_2} \leq k_j$$

for any $x_1, x_2 \in R$ and $x_1 \neq x_2$, and the matrix $-A + CK^{-1}$ is Lyapunov diagonally stable (LDS), then the network is globally asymptotically stable. $-A + CK^{-1}$ being LDS means that there exists a diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i > 0$ such that $\frac{1}{2}[(\alpha(-A + CK^{-1}))^T + \alpha(-A + CK^{-1})]$ is positive definite. This condition is obviously equivalent to the previous one that $\mathrm{diag}(\alpha_1 c_1/k_1, \ldots, \alpha_n c_n/k_n) + \frac{1}{2}(\alpha A + A^T \alpha)$ is negative definite, but it is applicable to more general networks. These two conditions are extensions of the above mentioned condition (vi), because the condition that the matrix $\frac{1}{2}[(\alpha(-A + CK^{-1}))^T + \alpha(-A + CK^{-1})]$ is positive definite is equivalent to that the matrix measure $\mu_2((\sqrt{\alpha CK^{-1}})^{-1}\alpha A(\sqrt{\alpha CK^{-1}})^{-1}) < 1$; where $\sqrt{\alpha CK^{-1}}$ stands for a diagonal matrix with diagonal entries that are square roots of diagonal entries of αCK^{-1}; when taking $\alpha_i = 1$, $c_i = k_i$, this condition reduces to condition (vi). Thus the above conditions (iii), (iv) and (vi), expressed by matrix measures, are three basic results. In this paper, among others, we will generalize, to some extent, these three conditions to delayed neural network cases.

It is not difficult to prove that under conditions (A1) and (A2) there exists at least one equilibrium for DNN [Eq. (1)] and that any solution of Eq. (1) is bounded on $[0, +\infty)$; see, for example, Lemmas 1 and 2 of Ref. [10] for details. Now suppose that $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is an equilibrium of Eq. (1); by applying the transformation $y_i(t) = x_i(t) - x_i^*$, $i = 1,2,\ldots,n$, Eq. (1) can be rewritten as

$$\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij}^0 g_j(y_j(t)) + \sum_{j=1}^{n} a_{ij}^\tau g_j(y_j(t-\tau_j)), \quad i = 1,2,\ldots,n \tag{4}$$

or

$$\dot{y}(t) = -Cy(t) + Ag(y(t)) + A^\tau g(y(t-\tau)),$$

where $y(t) = (y_1(t), \ldots, y_n(t))^T$, $g(y(t)) = [g_1(y_1(t)), \ldots, g_n(y_n(t))]^T$, $g(y(t-\tau)) = [g_1(y_1(t-\tau_1)), \ldots, g_n(y_n(t-\tau_n))]^T$, and $g_j(y_j(t)) \overset{\mathrm{def}}{=} f_j(y_j(t) + x_j^*) - f_j(x_j^*)$, $j = 1,2,\ldots,n$. According to the properties of f_j, g_j possesses the following properties.

(1) $|g_j(y_j)| \leq k_j|y_j|$.

(2) g_j is bounded and monotonic nondecreasing and $g_j(0) = 0$.

(3) $y_j g_j(y_j) \geq 0$, $g_j^2(y_j) \leq k_j y_j g_j(y_j)$, and $y_j g_j(y_j) \leq k_j y_j^2$ for any y_j.

The stability of Eq. (1) around x^* corresponds to that of Eq. (4) around the trivial equilibrium, so we just consider system (4). Generally speaking, an analysis of the dynamics of delayed neural networks is much more difficult than that of networks without delays, because the introduction of delays into a network makes the system of equations become infinite dimensional.

The differential difference equations (1) and (4) are categorized as retarded functional differential equations,

$$dx(t)/dt = f(t, x_t), \tag{5}$$

where $x(t) \in R^n$, $f: R \times C \to R^n$ is a functional defined on $R \times C$. $C \overset{\mathrm{def}}{=} C([-\tau, 0], R^n)$ stands for the Banach space of continuous functions mapping the interval $[-\tau,0]$ into R^n with the norm $\|\phi\|=\sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$, where $|\phi(\theta)|$ is the l_1 norm of R^n, i.e., $|\phi(\theta)|=\Sigma_{1\leq i\leq n}|\phi_i(\theta)|$. $x_t\in C$ is defined as $x_t(\theta)=x(t+\theta)$ for any $\theta\in[-\tau,0]$. For the functional differential equation (5), its initial condition is an element of C. Under some conditions on f (see Ref. [23]) for any given $\phi\in C$ and $t_0\in R$, there exists a unique solution of Eq. (5) with an initial condition (t_0,ϕ), which is simply the solution through (t_0,ϕ), denoted as $x(t_0,\phi)$. For a precise definition of the GAS of an equilibrium of a functional differential equation, see Ref. [23], here we just give a brief definition of the GES adopted from [23].

Definition [23]: For the retarded functional differential equation (5), suppose $x=0$ is its unique equilibrium, i.e., $f(t,0)=0$; then $x=0$ is said to be globally exponentially stable if there exist two numbers $B\geq 1$ and $\alpha>0$ such that, for any (t_0,ϕ), the solution $x(t_0,\phi)$ through (t_0,ϕ) satisfies

$$|x(t_0,\phi)|\leq B\|\phi\|e^{(-\alpha(t-t_0))}. \tag{6}$$

This definition defines the exponential stability based on the l_1 norm of R^n and its corresponding induced norm in C. The definition can also be given with respect to other norms. For example, Eq. (6) can be replaced with

$$|x(t_0,\phi)|_i\leq B\|\phi\|_i e^{(-\alpha(t-t_0))},\quad i=2,\infty, \tag{7}$$

where $|x|_2=(\Sigma_{i=1}^n|x_i|^2)^{1/2}$, $|x|_\infty=\max_{1\leq i\leq n}\{|x_i|\}$, and $\|\phi\|_i \overset{\text{def}}{=} \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|_i$. The equivalence of the three norms $l_i,i=1,2,\infty$ ensures the equivalence of definitions by these different norms.

II. MAIN RESULTS

Theorem 1. If there exists a diagonal matrix $P=\mathrm{diag}(p_1,p_2,\ldots,p_n)$ with positive numbers $p_i>0$, $i=1,\ldots,n$ such that

$$-CPK^{-1}+\frac{PA+A^TP}{2}+\|PA^\tau\|_{2,m}I \tag{8}$$

is negative definite, then network (4) [of course network (1)] is globally asymptotically stable, where $\|\cdot\|_{2,m}$ stands for either the spectral norm $\|\cdot\|_2$ or the Euclidean norm $\|\cdot\|_m$ of a square matrix.

Proof. For model (4), consider the following functional of $y_1,y_2,\ldots,y_n$:

$$V(y_1,y_2,\ldots,y_n)(t)=\sum_{i=1}^n p_i\int_0^{y_i(t)}g_i(s)\,ds+\frac{1}{2}\|PA^\tau\|_{2,m}\sum_{i=1}^n\int_{t-\tau_i}^t g_i^2(y_i(s))\,ds. \tag{9}$$

This is obviously a positive definite functional with respect to $y_1,y_2,\ldots,y_n$, that takes zero only at $(y_1,y_2,\ldots,y_n)^T=(0,0,\ldots,0)^T$, according to the properties of g_i's. Differentiating V with respect to time along the solution of Eq. (4), we have

$$\frac{dV}{dt}=\sum_{i=1}^n p_i\left\{g_i(y_i(t))\left(-c_iy_i(t)+\sum_{j=1}^n a_{ij}^0g_j(y_j(t))+\sum_{j=1}^n a_{ij}^\tau g_j(y_j(t-\tau_j))\right)+\frac{\|PA^\tau\|_{2,m}}{2}\sum_{i=1}^n[g_i^2(y_i(t))-g_i^2(y_i(t-\tau_i))]\right\}$$

$$=-\sum_{i=1}^n c_ip_iy_i(t)g_i(y_i(t))+\sum_{i=1}^n\sum_{j=1}^n p_ig_i(y_i(t))a_{ij}^0g_j(y_j(t))+\sum_{i=1}^n\sum_{j=1}^n p_ig_i(y_i(t))a_{ij}^\tau g_j(y_j(t-\tau_j))+\frac{\|PA^\tau\|_{2,m}}{2}\sum_{i=1}^n[g_i^2(y_i(t))-g_i^2(y_i(t-\tau_i))]$$

$$=-\sum_{i=1}^n c_ip_iy_i(t)g_i(y_i(t))+\sum_{i=1}^n\sum_{j=1}^n p_ig_i(y_i(t))a_{ij}^0g_j(y_j(t))+g(y(t))^TPA^\tau g(y(t-\tau))+\frac{\|PA^\tau\|_{2,m}}{2}\sum_{i=1}^n[g_i^2(y_i(t))-g_i^2(y_i(t-\tau_i))]$$

$$\leq-\sum_{i=1}^n c_ip_iy_i(t)g_i(y_i(t))+\sum_{i=1}^n\sum_{j=1}^n p_ig_i(y_i(t))a_{ij}^0g_j(y_j(t))+|g(y(t))^T|_2\cdot\|PA^\tau\|_{2,m}\cdot|g(y(t-\tau))|_2+\frac{\|PA^\tau\|_{2,m}}{2}\sum_{i=1}^n[g_i^2(y_i(t))-g_i^2(y_i(t-\tau_i))]$$

$$\leqslant -\sum_{i=1}^{n} c_i p_i y_i(t) g_i(y_i(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i g_i(y_i(t)) a_{ij}^0 g_j(y_j(t)) + \frac{\|PA^\tau\|_{2,m}}{2}\{[|g(y(t))^T|_2]^2 + [|g(y(t-\tau))|_2]^2\}$$
$$+ \frac{\|PA^\tau\|_{2,m}}{2}\sum_{i=1}^{n}[g_i^2(y_i(t)) - g_i^2(y_i(t-\tau_i))]$$
$$= -\sum_{i=1}^{n} c_i p_i y_i(t) g_i(y_i(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i g_i(y_i(t)) a_{ij}^0 g_j(y_j(t)) + \|PA^\tau\|_{2,m}[|g(y(t))|_2]^2$$
$$\leqslant -\sum_{i=1}^{n} \frac{c_i p_i}{k_i} g_i^2(y_i(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i g_i(y_i(t)) a_{ij}^0 g_j(y_j(t)) + \|PA^\tau\|_{2,m}(|g(y(t))|_2)^2$$
$$= -g(y(t))^T CPK^{-1} g(y(t)) + g(y(t))^T PA g(y(t)) + \|PA^\tau\|_{2,m} g(y(t))^T g(y(t))$$
$$= g(y(t))^T(-CPK^{-1} + PA + \|PA^\tau\|_{2,m} I) g(y(t))$$
$$= g(y(t))^T\left(-CPK^{-1} + \frac{PA + A^T P}{2} + \|PA^\tau\|_{2,m} I\right) g(y(t)). \tag{10}$$

The first inequality in Eq. (10) is based on the Cauchy-Schwarz inequality and the compatibility of matrix norms $\|\cdot\|_2$ and $\|\cdot\|_m$ with the Euclidean vector norm $|x|_2 \overset{\text{def}}{=} (\Sigma_{i=1}^n x_i^2)^{1/2}$. The second inequality is a result of the inequality $ab \leqslant \frac{1}{2}(a^2+b^2)$, and the third one is due to the property $g_j^2(y_j) \leqslant k_j y_j g_j(y_j)$. If $-CPK^{-1} + (PA + A^T P)/2 + \|PA^\tau\|_{2,m}$ is negative definite, then $dV/dt < 0$ at any $(y_1, y_2, \dots, y_n)^T \neq (0,0,\dots,0)^T$. This implies that V is a Lyapunov functional for Eq. (4), and that $(y_1, y_2, \dots, y_n)^T = (0,0,\dots,0)^T$ is the only globally asymptotically stable equilibrium. This completes the proof.

Remark 1. When $P = I$ and $C = K$, condition (8) turns to $(\|A^\tau\|_{2,m} - 1)I + (A + A^T)/2$, which is negative definite, and this is identical to

$$\lambda_{\max}\left\{(\|A^\tau\|_{2,m} - 1)I + \frac{A + A^T}{2}\right\}$$
$$= \|A^\tau\|_{2,m} - 1 + \lambda_{\max}\left(\frac{A + A^T}{2}\right) < 0,$$

that is,

$$\mu_2(A) + \|A^\tau\|_{2,m} < 1. \tag{11}$$

Condition (11) coincides with Theorem 4 of Ref. [6]. Arik and Tavsanoglu [13] [Theorem 1] give a condition which says that if (i) $-(A + A^T)$ is positive definite; (ii) $\|A^\tau\|_2 \leqslant 1$, then the DNN [Eq. (1)] is globally asymptotically stable. Since condition (i) $-(A + A^T)$ is positive definite is equivalent to that $\lambda_{\max}(A + A^T) < 0$, and this is $\mu_2(A) < 0$, so conditions (i) and (ii) together imply that $\mu_2(A) + \|A^\tau\|_2 < 1$; this is a part of condition (11), but when condition (11) is satisfied, conditions (i) and (ii) are not necessarily satisfied, and this means that our conditions (8) and (11) are more general than conditions (i) and (ii) of Ref. [13].

Theorem 2. Suppose there exists a diagonal matrix $P = \mathrm{diag}(p_1, p_2, \dots, p_n)$ with positive numbers $p_i > 0$, $i = 1, \dots, n$, such that

$$-CPK^{-1} + \frac{PA + A^T P}{2} + A^* \tag{12}$$

is negative definite. Then network (4) is globally asymptotically stable, where

$$A^* \overset{\text{def}}{=} \mathrm{diag}\left(\frac{1}{2}\sum_{j=1}^{n}(p_1|a_{1j}^\tau| + p_j|a_{j1}^\tau|), \frac{1}{2}\right.$$
$$\times \sum_{j=1}^{n}(p_2|a_{2j}^\tau| + p_j|a_{j2}^\tau|), \dots, \frac{1}{2}$$
$$\left.\times \sum_{j=1}^{n}(p_n|a_{nj}^\tau| + p_j|a_{jn}^\tau|)\right)$$

is a diagonal matrix.

Proof. For model (4), consider the following functional of $y_1, y_2, \dots, y_n$:

$$V(y_1, y_2, \dots, y_n)(t) = \sum_{i=1}^{n} p_i\left\{\int_0^{y_i(t)} g_i(s)\, ds \right.$$
$$\left. + \frac{1}{2}\sum_{j=1}^{n} |a_{ij}^\tau| \int_{t-\tau_j}^{t} g_j^2(y_j(s))\, ds\right\}. \tag{13}$$

This is obviously a positive definite functional with respect to $y_1, y_2, \dots, y_n$, that takes zero only at $(y_1, y_2, \dots, y_n)^T = (0,0,\dots,0)^T$, according to the properties of g_i's. Differentiating V with respect to time along the solution of Eq. (4), we have

$$
\begin{aligned}
\frac{dV}{dt} &= \sum_{i=1}^{n} p_i \left\{ g_i(y_i(t)) \left(-c_i y_i(t) + \sum_{j=1}^{n} a_{ij}^{0} g_j(y_j(t)) + \sum_{j=1}^{n} a_{ij}^{\tau} g_j(y_j(t-\tau_j)) \right) + \frac{1}{2} \sum_{j=1}^{n} |a_{ij}^{\tau}| [g_j^2(y_j(t)) - g_j^2(y_j(t-\tau_j))] \right\} \\
&= -\sum_{i=1}^{n} c_i p_i y_i(t) g_i(y_i(t)) + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i g_i(y_i(t)) a_{ij}^{0} g_j(y_j(t)) + \sum_{i=1}^{n} p_i \left\{ \sum_{j=1}^{n} g_i(y_i(t)) a_{ij}^{\tau} g_j(y_j(t-\tau_j)) \right. \\
&\quad \left. + \frac{1}{2} \sum_{j=1}^{n} |a_{ij}^{\tau}| [g_j^2(y_j(t)) - g_j^2(y_j(t-\tau_j))] \right\} \\
&\leqslant -\sum_{i=1}^{n} c_i p_i y_i(t) g_i(y_i(t)) + g(y(t))^T P A g(y(t)) + \sum_{i=1}^{n} p_i \left\{ \frac{1}{2} \sum_{j=1}^{n} |a_{ij}^{\tau}| [g_i^2(y_i(t)) + g_j^2(y_j(t-\tau_j))] \right. \\
&\quad \left. + \frac{1}{2} \sum_{j=1}^{n} |a_{ij}^{\tau}| [g_j^2(y_j(t)) - g_j^2(y_j(t-\tau_j))] \right\} \\
&= -\sum_{i=1}^{n} c_i p_i y_i(t) g_i(y_i(t)) + g(y(t))^T P A g(y(t)) + \sum_{i=1}^{n} \left\{ \frac{1}{2} \sum_{j=1}^{n} (p_i |a_{ij}^{\tau}| + p_j |a_{ji}^{\tau}|) g_i^2(y_i(t)) \right\} \\
&\leqslant -\sum_{i=1}^{n} \frac{c_i p_i}{k_i} g_i^2(y_i(t)) + g(y(t))^T P A g(y(t)) + \sum_{i=1}^{n} \left\{ \frac{1}{2} \sum_{j=1}^{n} (p_i |a_{ij}^{\tau}| + p_j |a_{ji}^{\tau}|) g_i^2(y_i(t)) \right\} \\
&= -g(y(t))^T C P K^{-1} g(y(t)) + g(y(t))^T P A g(y(t)) + g(y(t))^T A^* g(y(t)) \\
&= g(y(t))^T \{ -CPK^{-1} + PA + A^* \} g(y(t)) \\
&= g(y(t))^T \left\{ -CPK^{-1} + \frac{PA + A^T P}{2} + A^* \right\} g(y(t)).
\end{aligned} \tag{14}
$$

If the symmetric matrix $-CPK^{-1}+(PA+A^TP)/2+A^*$ is negative definite, then $dV/dt<0$ for any $(y_1,y_2,\ldots,y_n)^T \neq (0,0,\ldots,0)^T$. This implies that V is a Lyapunov functional of Eq. (4), and that $(0,0,\ldots,0)^T$ is the only globally asymptotically stable equilibrium. This completes the proof.

The condition that the symmetric matrix $-CPK^{-1}+(PA+A^TP)/2+A^*$ is negative definite is equivalent to

$$\lambda_{\max}\left\{ -CPK^{-1} + \frac{(PA+A^TP)}{2} + A^* \right\} < 0. \tag{15}$$

By using the fact that every eigenvalue of a symmetric matrix M is not greater than any measure of M,

$$\lambda_{\max}(M) \leqslant \mu_1(M) = \mu_\infty(M),$$

we obtain a more tractable(though more restrictive) stability condition.

Corollary 1. If $\mu_1(-CPK^{-1}+(PA+A^TP)/2+A^*)<0$, i.e.,

$$
\begin{aligned}
-\frac{c_i p_i}{k_i} + p_i a_{ii}^{0} + \frac{1}{2} \sum_{j \neq i} |p_i a_{ij}^{0} + p_j a_{ji}^{0}| \\
+ \frac{1}{2} \sum_{j=1}^{n} (p_i |a_{ij}^{\tau}| + p_j |a_{ji}^{\tau}|) < 0, \quad i = 1,2,\ldots,n,
\end{aligned} \tag{16}
$$

then network (4) is globally asymptotically stable.

Remark 2. Condition (15) is equivalent to

$$\mu_2(-CPK^{-1}+PA+A^*)<0. \tag{17}$$

When $P=I$ and $C=K$, this condition turns to

$$\mu_2(A+A^*)<1.$$

Since $\mu_2(A+A^*) \leqslant \mu_2(A)+\mu_2(A^*)$ and $\mu_2(A^*) = \frac{1}{2}\max_{1\leqslant i\leqslant n}\{\sum_{j=1}^{n}(|a_{ij}^{\tau}|+|a_{ji}^{\tau}|\}$, a slightly stronger criterion can be derived as

$$\mu_2(A) + \frac{1}{2} \max_{1\leqslant i\leqslant n} \left\{ \sum_{j=1}^{n} (|a_{ij}^{\tau}| + |a_{ji}^{\tau}|) \right\} < 1. \tag{18}$$

In addition, the inequality

$$
\begin{aligned}
\max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} (|a_{ij}^{\tau}| + |a_{ji}^{\tau}|) &\leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}^{\tau}| + \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ji}^{\tau}| \\
&= \|A^{\tau}\|_\infty + \|A^{\tau}\|_1
\end{aligned}
$$

leads to a further stronger sufficient condition for global asymptotic stability:

$$\mu_2(A)+\frac{1}{2}(\|A^\tau\|_\infty+\|A^\tau\|_1)<1. \tag{19}$$

Conditions (11), (17), and (19) show that Theorems 1 and 2 are generalization of condition (vi) for neural networks without delays to the case of DNN's. From conditions (11) and (19), some useful criteria for global asymptotic stability could be derived.

(Criterion 1) If the matrix A is skew symmetric, i.e., $A=-A^T$, and $\|A^\tau\|_{2,m}<1$ or $\|A^\tau\|_\infty+\|A^\tau\|_1<2$.

(Criterion 2) If the matrix $-(A+A^T)$ is positive definite, $\|A^\tau\|_\infty+\|A^\tau\|_1<2$.

Remark 3. When $P=I$ and $C=K$, condition (16) reduces to

$$a_{ii}^0+\frac{1}{2}\sum_{j\neq i}|a_{ij}^0+a_{ji}^0|+\frac{1}{2}\sum_{j=1}^n(|a_{ij}^\tau|+|a_{ji}^\tau|)<1,$$

$$i=1,2,\ldots,n.$$

This condition is more general than Cao's condition (ii) in Ref. [10].

Theorem 3. If there exist n positive numbers $p_1,p_2,\ldots,p_n$ and two positive numbers $r_1\in[0,1]$, $r_2\in[0,1]$ such that

$$\alpha_i\overset{\text{def}}{=}-c_ip_i+\frac{1}{2}\sum_{j\neq i}(p_i|a_{ij}^0|k_j^{2r_1}+p_j|a_{ji}^0|k_i^{2(1-r_1)})+\frac{1}{2}\sum_{j=1}^n(p_i|a_{ij}^\tau|k_j^{2r_2}+p_j|a_{ji}^\tau|k_i^{2(1-r_2)})<0,\quad i=1,2,\ldots,n, \tag{20}$$

and for all positive $a_{ii}^0>0$, $i\in\{1,2,\ldots,n\}$,

$$a_{ii}^0<-\frac{\alpha_i}{p_ik_i}, \tag{21}$$

then the trivial solution of network (4) is globally exponentially stable.

Proof. From conditions (20) and (21), there exists a small real number $\varepsilon>0$ such that

$$\alpha_i'\overset{\text{def}}{=}\left(-c_i+\frac{\varepsilon}{2}\right)p_i+\frac{1}{2}\sum_{j\neq i}(p_i|a_{ij}^0|k_j^{2r_1}+p_j|a_{ji}^0|k_i^{2(1-r_1)})+\frac{1}{2}\sum_{j=1}^n(p_i|a_{ij}^\tau|k_j^{2r_2}+p_j|a_{ji}^\tau|k_i^{2(1-r_2)}e^{\varepsilon\tau_i})<0,\quad i=1,2,\ldots,n, \tag{22}$$

and for all positive $a_{ii}^0>0$, $i\in\{1,2,\ldots,n\}$,

$$a_{ii}^0<-\frac{\alpha_i'}{p_ik_i}. \tag{23}$$

Consider the Lyapunov functional

$$V(y_1,y_2,\ldots,y_n)(t)=\sum_{i=1}^n p_i\left\{\frac{1}{2}y_i^2(t)e^{\varepsilon t}+\frac{1}{2}\sum_{j=1}^n|a_{ij}^\tau|\times k_j^{2(1-r_2)}\int_{t-\tau_j}^t y_j^2(s)e^{\varepsilon(s+\tau_j)}ds\right\}, \tag{24}$$

or rewritten as

$$V(t,\phi)=\sum_{i=1}^n p_i\left\{\frac{1}{2}\phi_i^2(0)e^{\varepsilon t}+\frac{1}{2}\sum_{j=1}^n|a_{ij}^\tau|k_j^{2(1-r_2)}\times\int_{t-\tau_j}^t\phi_j^2(s-t)e^{\varepsilon(s+\tau_j)}ds\right\}. \tag{25}$$

This is obviously a positive definite functional with respect to $y_1,y_2,\ldots,y_n$, that takes zero only at $(y_1,y_2,\ldots,y_n)^T=(0,0,\ldots,0)^T$, according to the properties of g_i's. Differentiating V with respect to time along the solution of Eq. (4), we have

$$\frac{dV}{dt}=\sum_{i=1}^n p_i\left\{y_i(t)\left(-c_iy_i(t)+\sum_{j=1}^n a_{ij}^0g_j(y_j(t))+\sum_{j=1}^n a_{ij}^\tau g_j(y_j(t-\tau_j))\right)e^{\varepsilon t}+\frac{1}{2}y_i^2(t)\varepsilon e^{\varepsilon t}+\frac{1}{2}\sum_{j=1}^n|a_{ij}^\tau|k_j^{2(1-r_2)}(y_j^2(t)e^{\varepsilon(t+\tau_j)}-y_j^2(t-\tau_j)e^{\varepsilon t})\right\}$$

$$=\sum_{i=1}^n p_i\left\{-c_iy_i^2(t)e^{\varepsilon t}+\frac{\varepsilon}{2}y_i^2(t)e^{\varepsilon t}+\sum_{j=1}^n y_i(t)a_{ij}^0g_j(y_j(t))e^{\varepsilon t}+\sum_{j=1}^n y_i(t)a_{ij}^\tau g_j(y_j(t-\tau_j))e^{\varepsilon t}+\frac{1}{2}\sum_{j=1}^n|a_{ij}^\tau|k_j^{2(1-r_2)}(y_j^2(t)e^{\varepsilon\tau_j}e^{\varepsilon t}-y_j^2(t-\tau_j)e^{\varepsilon t})\right\}$$

$$\leqslant\sum_{i=1}^{n} p_i\Bigg\{\left(-c_i+\frac{\varepsilon}{2}\right)y_i^2(t)e^{\varepsilon t}+a_{ii}^0 y_i(t)g_i(y_i(t))e^{\varepsilon t}+\sum_{j\neq i}|y_i(t)||a_{ij}^0|k_j|y_j(t)|e^{\varepsilon t}+\sum_{j=1}^{n}|y_i(t)||a_{ij}^{\tau}|k_j|y_j(t-\tau_j)|$$
$$\times e^{\varepsilon t}+\frac{1}{2}\sum_{j=1}^{n}|a_{ij}^{\tau}|k_j^{2(1-r_2)}(y_j^2(t)e^{\varepsilon\tau_j}e^{\varepsilon t}-y_j^2(t-\tau_j)e^{\varepsilon t})\Bigg\}$$

$$\leqslant\sum_{i=1}^{n} p_i\Bigg\{\left(-c_i+\frac{\varepsilon}{2}\right)y_i^2(t)e^{\varepsilon t}+a_{ii}^0 y_i(t)g_i(y_i(t))e^{\varepsilon t}+\frac{1}{2}\sum_{j\neq i}|a_{ij}^0|e^{\varepsilon t}(k_j^{2r_1}y_i^2(t)+k_j^{2(1-r_1)}y_j^2(t))$$
$$+\frac{1}{2}\sum_{j=1}^{n}|a_{ij}^{\tau}|e^{\varepsilon t}(k_j^{2r_2}y_i^2(t)+k_j^{2(1-r_2)}y_j^2(t-\tau_j))+\frac{1}{2}\sum_{j=1}^{n}|a_{ij}^{\tau}|k_j^{2(1-r_2)}(y_j^2(t)e^{\varepsilon\tau_j}e^{\varepsilon t}-y_j^2(t-\tau_j)e^{\varepsilon t})\Bigg\}$$

$$=\sum_{i=1}^{n}\Bigg\{a_{ii}^0 p_i y_i(t)g_i(y_i(t))e^{\varepsilon t}+\Bigg[\left(-c_i+\frac{\varepsilon}{2}\right)p_i+\frac{1}{2}\sum_{j\neq i}(p_i|a_{ij}^0|k_j^{2r_1}+p_j|a_{ji}^0|k_i^{2(1-r_1)})+\frac{1}{2}\sum_{j=1}^{n}(p_i|a_{ij}^{\tau}|k_j^{2r_2}$$
$$+p_j|a_{ji}^{\tau}|k_i^{2(1-r_2)}e^{\varepsilon\tau_i})\Bigg]y_i^2(t)e^{\varepsilon t}\Bigg\}$$

$$=\sum_{i=1}^{n}\{a_{ii}^0 p_i y_i(t)g_i(y_i(t))+\alpha_i' y_i^2(t)\}e^{\varepsilon t}. \tag{26}$$

From Eq. (22), $\alpha_i'<0$; thus, according to the properties of the functions g_i,

$$\alpha_i' y_i^2(t)\leqslant\frac{\alpha_i'}{k_i}y_i(t)g_i(y_i(t)).$$

Therefore, it follows from Eq. (26) that

$$\frac{dV}{dt}\leqslant\sum_{i=1}^{n}\left(\frac{\alpha_i'}{k_i}+a_{ii}^0 p_i\right)y_i(t)g_i(y_i(t))e^{\varepsilon t}.$$

It again follows from conditions (22) and (23) that $\alpha_i'/k_i+a_{ii}^0 p_i<0$ for all $i=1,2,\ldots,n$, together with the properties of g_i, we have

$$\frac{dV}{dt}\leqslant\sum_{i=1}^{n}\left(\frac{\alpha_i'}{k_i^2}+\frac{a_{ii}^0 p_i}{k_i}\right)g_i^2(y_i(t)))e^{\varepsilon t}\leqslant-\beta_1\sum_{i=1}^{n}g_i^2(y_i(t))e^{\varepsilon t},$$

where $\beta_1=\min_{1\leqslant i\leqslant n}\{-\alpha_i'/k_i^2-a_{ii}^0 p_i/k_i\}>0$. From this, we have

$$V(t)\leqslant V(0),\quad\text{for all}\quad t>0. \tag{27}$$

The construction of the Lyapunov functional implies that

$$\frac{1}{2}\min_{1\leqslant i\leqslant n}\{p_i\}e^{\varepsilon t}\sum_{i=1}^{n}y_i^2(t)\leqslant\sum_{i=1}^{n}\frac{1}{2}p_i y_i^2(t)e^{\varepsilon t}\leqslant V(t), \tag{28}$$

and from Eq. (25), we have

$$V(0)=\sum_{i=1}^{n}p_i\Bigg\{\frac{1}{2}\phi_i^2(0)+\frac{1}{2}\sum_{j=1}^{n}|a_{ij}^{\tau}|k_j^{2(1-r_2)}\int_{-\tau_j}^{0}\phi_j^2(s)e^{\varepsilon(s+\tau_j)}ds\Bigg\}$$
$$\leqslant\frac{1}{2}\max_{1\leqslant i\leqslant n}\{p_i\}\sum_{i=1}^{n}\phi_i^2(0)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_i|a_{ij}^{\tau}|k_j^{2(1-r_2)}e^{\varepsilon\tau_j}\int_{-\tau_j}^{0}\phi_j^2(s)ds$$

$$\leqslant \frac{1}{2}\max_{1\leqslant i\leqslant n}\{p_i\}\|\phi\|_2^2+\sum_{j=1}^{n}\left\{\frac{1}{2}\sum_{i=1}^{n}p_i|a_{ij}^{\tau}|k_j^{2(1-r_2)}e^{\varepsilon\tau_j}\right\}\int_{-\tau}^{0}\phi_j^2(s)ds$$

$$\leqslant \frac{1}{2}\max_{1\leqslant i\leqslant n}\{p_i\}\|\phi\|_2^2+\max_{1\leqslant j\leqslant n}\left\{\frac{1}{2}\sum_{i=1}^{n}p_i|a_{ij}^{\tau}|k_j^{2(1-r_2)}e^{\varepsilon\tau_j}\right\}\sum_{j=1}^{n}\int_{-\tau}^{0}\phi_j^2(s)ds$$

$$=\frac{1}{2}\max_{1\leqslant i\leqslant n}\{p_i\}\|\phi\|_2^2+\max_{1\leqslant j\leqslant n}\left\{\frac{1}{2}\sum_{i=1}^{n}p_i|a_{ij}^{\tau}|k_j^{2(1-r_2)}e^{\varepsilon\tau_j}\right\}\int_{-\tau}^{0}\sum_{j=1}^{n}\phi_j^2(s)ds$$

$$\leqslant\left(\frac{1}{2}\max_{1\leqslant i\leqslant n}\{p_i\}+\max_{1\leqslant j\leqslant n}\left\{\frac{1}{2}\sum_{i=1}^{n}p_i|a_{ij}^{\tau}|k_j^{2(1-r_2)}e^{\varepsilon\tau_j}\tau\right\}\right)\|\phi\|_2^2. \tag{29}$$

Combining inequalities (27), (28), and (29), we have

$$\|y(t)\|_2=\left(\sum_{i=1}^{n}y_i^2(t)\right)^{1/2}\leqslant B\|\phi\|_2e^{-(\varepsilon/2)t},$$

where

$$B=\max\left\{1,\left[(\min_{1\leqslant i\leqslant n}\{p_i\})^{-1}\left(\max_{1\leqslant i\leqslant n}\{p_i\}+\max_{1\leqslant j\leqslant n}\left\{\sum_{i=1}^{n}p_i|a_{ij}^{\tau}|k_j^{2(1-r_2)}e^{\varepsilon\tau_j}\tau\right\}\right)\right]^{1/2}\right\}.$$

This finishes the proof of the theorem.

Remark 4. The sufficient condition in theorem 3 is equivalent to

$$\frac{1}{c_i}\left\{2(a_{ii}^0)^+k_i+\sum_{j\neq i}\left(|a_{ij}^0|k_j^{2r_1}+\frac{p_j}{p_i}|a_{ji}^0|k_i^{2(1-r_1)}\right)+\sum_{j=1}^{n}\left(|a_{ij}^{\tau}|k_j^{2r_2}+\frac{p_j}{p_i}|a_{ji}^{\tau}|k_i^{2(1-r_2)}\right)\right\}<2,\quad i=1,2,\ldots,n. \tag{30}$$

When taking all $p_i=1$, this condition turns to

$$\frac{1}{c_i}\left\{2(a_{ii}^0)^+k_i+\sum_{j\neq i}(|a_{ij}^0|k_j^{2r_1}+|a_{ji}^0|k_i^{2(1-r_1)})+\sum_{j=1}^{n}(|a_{ij}^{\tau}|k_j^{2r_2}+|a_{ji}^{\tau}|k_i^{2(1-r_2)})\right\}<2,\quad i=1,2,\ldots,n. \tag{31}$$

Condition (31) corresponds to the result of Theorem 1 in Ref. [11], but is less conservative than that since $(a_{ii}^0)^+\leqslant|a_{ii}^0|$. Here global exponential stability is ensured.

Theorem 4. If there exist n positive real numbers $p_i>0$, $i=1,2,\ldots,n$ such that

$$\max_{1\leqslant i\leqslant n}\left\{-\frac{c_i}{k_i}+a_{ii}^0+\sum_{j\neq i}\frac{p_j}{p_i}|a_{ji}^0|+\sum_{j=1}^{n}\frac{p_j}{p_i}|a_{ji}^{\tau}|\right\}<0, \tag{32}$$

then the delayed neural network (4) is globally exponentially stable.

Proof. If condition (32) holds, there exists a positive number $\varepsilon>0$ such that $\varepsilon<c_i$, $i=1,2,\ldots,n$ and

$$\max_{1\leqslant i\leqslant n}\left\{\frac{\varepsilon-c_i}{k_i}+a_{ii}^0+\sum_{j\neq i}\frac{p_j}{p_i}|a_{ji}^0|+\sum_{j=1}^{n}\frac{p_j}{p_i}|a_{ji}^{\tau}|e^{\varepsilon\tau_i}\right\}<0. \tag{33}$$

For any initial condition $\phi\in C$, suppose the solution through ϕ is $y(t)=(y_1(t),\ldots,y_n(t))^T$, define a Lyapunov functional as

$$V(y_1,y_2,\ldots,y_n)(t)=\sum_{i=1}^{n}p_i\left\{|y_i(t)|e^{\varepsilon t}+\sum_{j=1}^{n}|a_{ij}^{\tau}|\int_{t-\tau_j}^{t}|g_j(y_j(s))|e^{\varepsilon(s+\tau_j)}ds\right\}, \tag{34}$$

or rewrite in terms of ϕ as

$$V(t,\phi)=\sum_{i=1}^{n}p_i\left\{|\phi_i(0)|e^{\varepsilon t}+\sum_{j=1}^{n}|a_{ij}^{\tau}|\int_{t-\tau_j}^{t}|g_j(\phi_j(s-t))|e^{\varepsilon(s+\tau_j)}ds\right\}. \tag{35}$$

Obviously, $V(y_1,y_2,\ldots,y_n)(t)>0$ for $(y_1,y_2,\ldots,y_n)\neq(0,0,\ldots,0)$ and $V(y_1,y_2,\ldots,y_n)(t)=0$ only at $(y_1,y_2,\ldots,y_n)=(0,0,\ldots,0)$. We calculate the upper right Dini derivative of V along the solution of Eq. (4) and estimating its right hand side we have

$$\begin{aligned}
D^+V=&\sum_{i=1}^n p_i\Bigg\{\varepsilon|y_i(t)|e^{\varepsilon t}+\sigma(y_i,dy_i/dt)\Bigg(-c_iy_i(t)+\sum_{j=1}^n a_{ij}^0 g_j(y_j(t))+\sum_{j=1}^n a_{ij}^\tau g_j(y_j(t-\tau_j))\Bigg)e^{\varepsilon t}\\
&+\sum_{j=1}^n |a_{ij}^\tau|[|g_j(y_j(t))|e^{\varepsilon(t+\tau_j)}-|g_j(y_j(t-\tau_j))|e^{\varepsilon t}]\Bigg\}\\
=&\sum_{i=1}^n p_i\Bigg\{a_{ii}^0 g_i(y_i(t))\sigma(y_i,dy_i/dt)e^{\epsilon t}+\sum_{j\neq i} a_{ij}^0\sigma(y_i,dy_i/dt\Bigg)g_j(y_j(t))e^{\varepsilon t}+\sum_{j=1}^n a_{ij}^\tau\sigma(y_i,dy_i/dt)g_j(y_j(t-\tau_j))\\
&\times e^{\varepsilon t}+\varepsilon|y_i(t)|e^{\varepsilon t}-c_i|y_i(t)|e^{\epsilon t}+\sum_{j=1}^n |a_{ij}^\tau||g_j(y_j(t))|e^{\varepsilon(t+\tau_j)}-\sum_{j=1}^n |a_{ij}^\tau||g_j(y_j(t-\tau_j))|e^{\varepsilon t}\\
\leqslant&\sum_{i=1}^n p_i\Bigg\{\frac{\varepsilon-c_i}{k_i}|g_i(y_i(t))|e^{\varepsilon t}+a_{ii}^0|g_i(y_i(t))|e^{\varepsilon t}+\sum_{j\neq i}|a_{ij}^0||g_j(y_j(t))|e^{\varepsilon t}+\sum_{j=1}^n |a_{ij}^\tau||g_j(y_j(t-\tau_j))|e^{\varepsilon t}\\
&+\sum_{j=1}^n |a_{ij}^\tau||g_j(y_j(t))|e^{\varepsilon(t+\tau_j)}-\sum_{j=1}^n |a_{ij}^\tau||g_j(y_j(t-\tau_j))|e^{\varepsilon t}\Bigg\}\\
=&\sum_{i=1}^n p_i\Bigg\{\Bigg(\frac{\varepsilon-c_i}{k_i}+a_{ii}^0\Bigg)|g_i(y_i(t))|e^{\varepsilon t}+\sum_{j\neq i}|a_{ij}^0||g_j(y_j(t))|e^{\varepsilon t}+\sum_{j=1}^n |a_{ij}^\tau||g_j(y_j(t))|e^{\varepsilon(t+\tau_j)}\Bigg\}\\
=&\sum_{i=1}^n p_i\Bigg\{\frac{\varepsilon-c_i}{k_i}+a_{ii}^0+\sum_{j\neq i}\Bigg(\frac{p_j}{p_i}|a_{ji}^0|\Bigg)+\sum_{j=1}^n\Bigg(\frac{p_j}{p_i}|a_{ji}^\tau|e^{\varepsilon\tau_i}\Bigg)\Bigg\}|g_i(y_i(t))|e^{\varepsilon t}\\
\leqslant&-\beta_2\sum_{i=1}^n |g_i(y_i(t))|e^{\varepsilon t},
\end{aligned}\tag{36}$$

where, by inequality (33),

$$\beta_2\overset{\text{def}}{=}\min_{1\leqslant i\leqslant n}\Bigg\{p_i\Bigg(\frac{c_i-\varepsilon}{k_i}-a_{ii}^0\Bigg)-\sum_{j\neq i}|a_{ji}^0|p_j-\sum_{j=1}^n|a_{ji}^\tau|p_je^{\varepsilon\tau_i}\Bigg\}>0.$$

In the estimation of Eq. (36), the facts that $\sigma(y_i,dy_i/dt)y_i=|y_i|$ and $\sigma(y_i,dy_i/dt)g_i(y_i)=|g_i(y_i)|$ have been used. The first inequality is due to the fact that $\varepsilon<c_i$ and $|g_i(y_i(t))|\leqslant k_i|y_i(t)|$.

It follows from Eq. (36) that

$$V(t)\leqslant V(0),\quad\text{for}\quad t>0,\tag{37}$$

and from Eq. (34) that

$$\min_{1\leqslant i\leqslant n}\{p_i\}e^{\varepsilon t}\sum_{i=1}^n |y_i(t)|\leqslant\sum_{i=1}^n p_i|y_i(t)|e^{\varepsilon t}\leqslant V(t).\tag{38}$$

According to Eq. (35),

$$\begin{aligned}
V(0)=&\sum_{i=1}^n p_i\Bigg\{|\phi_i(0)|+\sum_{j=1}^n|a_{ij}^\tau|\int_{-\tau_j}^0|g_j(\phi_j(s))|e^{\varepsilon(s+\tau_j)}ds\Bigg\}\\
\leqslant&\max_{1\leqslant i\leqslant n}\{p_i\}\sum_{i=1}^n|\phi_i(0)|+\sum_{i=1}^n\sum_{j=1}^n p_i|a_{ij}^\tau|\int_{-\tau_j}^0 k_j|\phi_j(s)|e^{\varepsilon\tau_j}ds
\end{aligned}$$

$$\leq \max_{1\leq i\leq n}\{p_i\}\|\phi\| + \sum_{j=1}^{n}\left\{\sum_{i=1}^{n} p_i|a_{ij}^{\tau}|k_j e^{\varepsilon\tau_j}\right\}\int_{-\tau_j}^{0}|\phi_j(s)|ds$$

$$\leq \max_{1\leq i\leq n}\{p_i\}\|\phi\| + \max_{1\leq j\leq n}\left\{\sum_{i=1}^{n} p_i|a_{ij}^{\tau}|k_j e^{\varepsilon\tau_j}\right\}\int_{-\tau}^{0}\sum_{j=1}^{n}|\phi_j(s)|ds$$

$$\leq\left(\max_{1\leq i\leq n}\{p_i\} + \max_{1\leq j\leq n}\left\{\sum_{i=1}^{n} p_i|a_{ij}^{\tau}|\tau k_j e^{\varepsilon\tau_j}\right\}\right)\|\phi\|. \tag{39}$$

Combining Eqs. (37), (38) and (39), we obtain

$$\sum_{i=1}^{n}|y_i(t)|\leq B\|\phi\|e^{-\varepsilon t}, \tag{40}$$

where

$$B=\max\left\{(\min_{1\leq i\leq n}\{p_i\})^{-1}\left(\max_{1\leq i\leq n}\{p_i\} + \max_{1\leq j\leq n}\left\{\sum_{i=1}^{n} p_i|a_{ij}^{\tau}|\tau k_j e^{\varepsilon\tau_j}\right\}\right),1\right\} \geq 1. \tag{41}$$

This has proven the theorem.

Remark 5. Condition (32) can be rewritten as $\mu_1(PAP^{-1}C^{-1}K)+\|PA^{\tau}P^{-1}C^{-1}K\|_1<1$, where $P=\mathrm{diag}(p_1,p_2,\ldots,p_n)$ is a diagonal matrix with diagonal entries as $p_1,p_2,\ldots,p_n$. If all $c_i=k_i$ in network (1), i.e., $C=K$, Eq. (32) is equivalent to $\mu_1(PAP^{-1})+\|PA^{\tau}P^{-1}\|_1<1$. When $P=I$, the identity matrix of size n, this condition further reduces to $\mu_1(A)+\|A^{\tau}\|<1$. This implies that condition (32) is an extension of above condition (iv) for neural networks without delay to the delayed case.

Remark 6. Zhang Yi [2] considered a special case of a network model [Eq. (1)] when ignoring the intraneural signal transmission delays, which is a special case of our model (1) when $a_{ij}^0=0$, for any $i\neq j$, $a_{ii}^{\tau}=0$ for any i and $C=K=I$. Sufficient conditions ensuring global exponential stability were established by the author as (in our notations) $a_{ii}^0<0$, $i=1,2,\ldots,n$ and $\mu_1(P(A+A^{\tau})P^{-1})<0$. These conditions are equivalent to $a_{ii}^0<0$ and

$$a_{ii}^0+\sum_{j\neq i}\frac{p_j}{p_i}|a_{ji}^{\tau}|<0.$$

Applying Eq. (32), our criterion can be expressed as

$$\max_{1\leq i\leq n}\left\{a_{ii}^0+\sum_{j\neq i}\frac{p_j}{p_i}|a_{ji}^{\tau}|\right\}<1.$$

It is obvious that our condition is weaker than Zhang Yi's. The authors of Ref. [3] investigated a special case of Eq. (1) when $A=0$. Without assuming that $\tau_i=0$, for $i=1,2,\ldots,n$, they proposed a condition for global asymptotic stability that could be rewritten in our notations as $\|A^{\tau}C^{-1}K\|_1<1$; this condition coincides with ours when $A=0$ and $P=I$.

Theorem 5. Suppose the condition

$$(a_{ii}^0)^+k_i+\sum_{j\neq i}|a_{ij}^0|k_j+\sum_{j=1}^{n}|a_{ij}^{\tau}|k_j<c_i,\quad i=1,2,\ldots,n, \tag{42}$$

holds. Then the delayed neural network [Eq. (4)] is globally exponentially stable.

Proof. If condition (42) holds, there must exist a positive real number ε such that

$$-c_i+\varepsilon+(a_{ii}^0)^+k_i+\sum_{j\neq i}|a_{ij}^0|k_j+\sum_{j=1}^{n}|a_{ij}^{\tau}|k_j e^{\varepsilon\tau_j}<0,\quad i=1,2,\ldots,n. \tag{43}$$

Suppose $y(t)=(y_1(t),y_2(t),\ldots,y_n(t))^T\in R^n$ is a solution of Eq. (4) with an initial function $\phi\in C$. For any $t>0$, we define $|y_{i(t)}(t)|=\max_{1\leq i\leq n}|y_i(t)|$, i.e., $1\leq i(t)\leq n$ is the index of the component of the solution vector $y(t)$ at which the maximum value of $|y_i(t)|$ is achieved. We construct a Lyapunov functional for network (4) as

$$V(y_1,y_2,\ldots,y_n)(t)=|y_{i(t)}(t)|e^{\varepsilon t}+\sum_{j=1}^{n}|a_{i(t)j}^{\tau}|\times\int_{t-\tau_j}^{t}|g_j(y_j(s))|e^{\varepsilon(s+\tau_j)}ds, \tag{44}$$

or rewrite with respect to ϕ as

$$V(t,\phi)=|\phi_{i(t)}(0)|e^{\varepsilon t}+\sum_{j=1}^{n}|a_{i(t)j}^{\tau}|\times\int_{t-\tau_j}^{t}|g_j(\phi_j(s-t))|e^{\varepsilon(s+\tau_j)}ds. \tag{45}$$

Obviously, V is bounded and for any $(y_1,y_2,\ldots,y_n)\neq(0,0,\ldots,0)$, $V(y_1,y_2,\ldots,y_n)(t)>0$. Calculating the upper right Dini derivative of Eq. (44) along the solution $y(t)$ of Eq. (4), we have

$$D^+V=\sigma(y_{i(t)},dy_{i(t)}/dt)\left(-c_{i(t)}y_{i(t)}(t)+\sum_{j=1}^{n} a_{i(t)j}^{0}g_j(y_j(t))+\sum_{j=1}^{n} a_{i(t)j}^{\tau}g_j(y_j(t-\tau_j))\right)e^{\varepsilon t}+\varepsilon|y_{i(t)}(t)|e^{\varepsilon t}$$
$$+\sum_{j=1}^{n} |a_{i(t)j}^{\tau}|(|g_j(y_j(t))|e^{\varepsilon(t+\tau_j)}-|g_j(y_j(t-\tau_j))|e^{\varepsilon t})$$
$$=-c_{i(t)}|y_{i(t)}(t)|e^{\varepsilon t}+a_{i(t)i(t)}^{0}g_{i(t)}(y_{i(t)}(t))\sigma(y_{i(t)},dy_{i(t)}/dt)e^{\varepsilon t}+\varepsilon|y_{i(t)}(t)|e^{\varepsilon t}$$
$$+\sum_{j\neq i(t)} a_{i(t)j}^{0}\sigma(y_{i(t)},dy_{i(t)}/dt)g_j(y_j(t))e^{\varepsilon t}+\sum_{j=1}^{n} a_{i(t)j}^{\tau}\sigma(y_{i(t)},dy_{i(t)}/dt)g_j(y_j(t-\tau_j))e^{\varepsilon t}$$
$$+\sum_{j=1}^{n} |a_{i(t)j}^{\tau}|[|g_j(y_j(t))|e^{\varepsilon(t+\tau_j)}-|g_j(y_j(t-\tau_j))|e^{\varepsilon t}]$$
$$\leqslant(-c_{i(t)}+\varepsilon)|y_{i(t)}(t)|e^{\varepsilon t}+a_{i(t)i(t)}^{0}|g_{i(t)}(y_{i(t)}(t))|e^{\varepsilon t}+\sum_{j\neq i(t)} |a_{i(t)j}^{0}||g_j(y_j(t))|e^{\varepsilon t}+\sum_{j=1}^{n} |a_{i(t)j}^{\tau}||g_j(y_j(t-\tau_j))|e^{\varepsilon t}$$
$$+\sum_{j=1}^{n} |a_{i(t)j}^{\tau}||g_j(y_j(t))|e^{\varepsilon(t+\tau_j)}-\sum_{j=1}^{n} |a_{i(t)j}^{\tau}||g_j(y_j(t-\tau_j))|e^{\varepsilon t}$$
$$=\left\{(-c_{i(t)}+\varepsilon)|y_{i(t)}(t)|+a_{i(t)i(t)}^{0}|g_{i(t)}(y_{i(t)}(t))|+\sum_{j\neq i(t)} |a_{i(t)j}^{0}||g_j(y_j(t))|+\sum_{j=1}^{n} |a_{i(t)j}^{\tau}||g_j(y_j(t))|e^{\varepsilon\tau_j}\right\}e^{\varepsilon t}$$
$$\leqslant\left\{(-c_{i(t)}+\varepsilon)|y_{i(t)}(t)|+(a_{i(t)i(t)}^{0})^{+}k_{i(t)}|y_{i(t)}(t)|+\sum_{j\neq i(t)} |a_{i(t)j}^{0}|k_j|y_j(t)|+\sum_{j=1}^{n} |a_{i(t)j}^{\tau}|k_j|y_j(t)|e^{\varepsilon\tau_j}\right\}e^{\varepsilon t}$$
$$\leqslant\left\{-c_{i(t)}+\varepsilon+(a_{i(t)i(t)}^{0})^{+}k_{i(t)}+\sum_{j\neq i(t)} |a_{i(t)j}^{0}|k_j+\sum_{j=1}^{n} |a_{i(t)j}^{\tau}|k_je^{\varepsilon\tau_j}\right\}|y_{i(t)}(t)|e^{\varepsilon t}$$
$$\leqslant-\beta_3|y_{i(t)}(t)|e^{\varepsilon t}, \tag{46}$$

where

$$\beta_3\overset{\text{def}}{=}\min_{1\leqslant i\leqslant n}\left\{c_i-\varepsilon-(a_{ii}^{0})^{+}k_i-\sum_{j\neq i} |a_{ij}^{0}|k_j-\sum_{j=1}^{n} |a_{ij}^{\tau}|k_je^{\varepsilon\tau_j}\right\}.$$

According to condition (43), $\beta_3>0$. It follows from Eq. (46) that

$$V(t)\leqslant V(0)\quad\text{for}\quad t>0. \tag{47}$$

From Eq. (45), we have

$$V(0)=|\phi_{i(0)}(0)|+\sum_{j=1}^{n} |a_{i(0)j}^{\tau}|\int_{-\tau_j}^{0}|g_j(\phi_j(s))|e^{\varepsilon(s+\tau_j)}ds$$
$$\leqslant\|\phi\|_\infty+\sum_{j=1}^{n} |a_{i(0)j}^{\tau}|k_je^{\varepsilon\tau}\int_{-\tau}^{0}|\phi_j(s)|ds$$
$$\leqslant\left(1+\sum_{j=1}^{n} |a_{i(0)j}^{\tau}|k_je^{\varepsilon\tau}\tau\right)\|\phi\|_\infty, \tag{48}$$

and Eq. (44) implies that

$$|y_{i(t)}(t)|e^{\varepsilon t}\leqslant V(t). \tag{49}$$

Combining Eqs. (47), (48), and (49), we obtain

$$|y(t)|_\infty=\max_{1\leqslant i\leqslant n}|y_i(t)|=y_{i(t)}(t)$$
$$\leqslant\left(1+\sum_{j=1}^{n} |a_{i(0)j}^{\tau}|k_je^{\varepsilon\tau}\tau\right)\|\phi\|_\infty e^{-\varepsilon t}. \tag{50}$$

This implies that the solution $y(t)$ converges, with respect to the norm $|\cdot|_\infty$ of R^n, to the only equilibrium $y=0$ at an exponential rate. The proof of the theorem is thus completed.

Corollary 2. When the activation functions of all neurons are identical, i.e., $g_i=g$ in Eq. (4), and further g is odd symmetry, i.e., $g(-x)=-g(x)$, then a global exponential stability criterion is

$$a_{ii}^{0}+\sum_{j\neq i} |a_{ij}^{0}|+\sum_{j=1}^{n} |a_{ij}^{\tau}|<\frac{c_i}{k_i},\quad i=1,2,\ldots,n. \tag{51}$$

Proof. If Eq. (51) holds, there must exist a small positive real number $\varepsilon>0$ such that $\varepsilon<c_i$ for all $i=1,2,\ldots,n$, and

$$\frac{(-c_i+\varepsilon)}{k_i}+a_{ii}^0+\sum_{j\neq i}|a_{ij}^0|+\sum_{j=1}^{n}|a_{ij}^\tau|e^{\varepsilon\tau_j}<0,$$

$$i=1,2,\ldots,n. \tag{52}$$

Note that if g is odd symmetry, then for $i(t)$ defined by $|y_{i(t)}(t)|=\max_{1\leqslant i\leqslant n}|y_i(t)|$, $|g(y_{i(t)}(t))|=\max_{1\leqslant i\leqslant n}|g(y_i(t))|$ also holds. Defining the same Lyapunov functional as in Theorem 5 and following a similar estimation procedure, we obtain

$$\begin{aligned}D^+V&\leqslant\left\{(-c_{i(t)}+\varepsilon)|y_{i(t)}(t)|+a_{i(t)i(t)}^0|g(y_{i(t)}(t))|+\sum_{j\neq i(t)}|a_{i(t)j}^0||g(y_j(t))|+\sum_{j=1}^{n}|a_{i(t)j}^\tau||g(y_j(t))|e^{\varepsilon\tau_j}\right\}e^{\varepsilon t}\\&\leqslant\left\{\frac{-c_{i(t)}+\varepsilon}{k_i}|g(y_{i(t)}(t))|+a_{i(t)i(t)}^0|g(y_{i(t)}(t))|+\sum_{j\neq i(t)}|a_{i(t)j}^0||g(y_j(t))|+\sum_{j=1}^{n}|a_{i(t)j}^\tau||g(y_j(t))|e^{\varepsilon\tau_j}\right\}e^{\varepsilon t}\\&\leqslant\left\{\frac{-c_{i(t)}+\varepsilon}{k_i}+a_{i(t)i(t)}^0+\sum_{j\neq i(t)}|a_{i(t)j}^0|+\sum_{j=1}^{n}|a_{i(t)j}^\tau|e^{\varepsilon\tau_j}\right\}|g(y_{i(t)}(t))|e^{\varepsilon t}\\&\leqslant-\beta_4|g(y_{i(t)}(t))|e^{\varepsilon t},\end{aligned} \tag{53}$$

where, by inequality (52),

$$\beta_4\overset{\text{def}}{=}\min_{1\leqslant i\leqslant n}\left\{\frac{c_i-\varepsilon}{k_i}-a_{ii}^0-\sum_{j\neq i}|a_{ij}^0|-\sum_{j=1}^{n}|a_{ij}^\tau|e^{\varepsilon\tau_j}\right\}>0.$$

The remainder of the proof is identical to that of theorem 5.

Remark 7. Condition (51) can be represented in matrix measures and norms as: $\mu_\infty(C^{-1}KA)+\|C^{-1}KA^\tau\|\infty<1$. When $C=K$, the condition becomes $\mu_\infty(A)+\|A^\tau\|_\infty<1$. This indicates that condition (51) is a generalized version of condition (iii) [17] for neural networks without delays. A more general version of Theorem 5 and Corollary 2 are stated in the Corollary 3.

Corollary 3. If there exist n positive real numbers $p_1,p_2,\ldots,p_n$ such that

$$-c_ip_i+(a_{ii}^0)^+k_ip_i+\sum_{j\neq i}|a_{ij}^0|k_jp_j+\sum_{j=1}^{n}|a_{ij}^\tau|k_jp_j<0,$$

$$i=1,2,\ldots,n, \tag{54}$$

or further when all activation functions are identical and odd symmetry, and

$$-c_ip_i+a_{ii}^0k_ip_i+\sum_{j\neq i}|a_{ij}^0|k_ip_j+\sum_{j=1}^{n}|a_{ij}^\tau|k_ip_j<0,$$

$$i=1,2,\ldots,n, \tag{55}$$

then neural network (4) [and thus network (1)] is globally exponentially stable.

Proof. Transforming Eq. (4) with $y_i=p_iz_i$, this turns to

$$\frac{dz_i(t)}{dt}=-c_iz_i(t)+\sum_{j=1}^{n}a_{ij}^0\frac{1}{p_i}g_j(p_jz_j(t))+\sum_{j=1}^{n}a_{ij}^\tau\frac{1}{p_i}g_j(p_jz_j(t-\tau_j)),\quad i=1,2,\ldots,n. \tag{56}$$

If one lets $h_j(z)=(1/p_j)g_j(p_jz)$, then Eq. (56) can be rewritten as

$$\frac{dz_i(t)}{dt}=-c_iz_i(t)+\sum_{j=1}^{n}a_{ij}^0\frac{p_j}{p_i}h_j(z_j(t))+\sum_{j=1}^{n}a_{ij}^\tau\frac{p_j}{p_i}h_j(z_j(t-\tau_j)),\quad i=1,2,\ldots,n. \tag{57}$$

Since $dh_j/dz|_z=dg_j/dy|_{p_jz}$, functions h_j have the same properties as g_j. Hence applying Theorem 5 and Corollary 2 to Eq. (57), respectively, we can obtain conditions (54) and (55) for system (57). Moreover, systems (4) and (57) have same stability properties. This finishes the proof of the corollary.

Remark 8. Condition (55) can be expressed in another way as

$$\mu_\infty(P^{-1}C^{-1}KAP)+\|P^{-1}C^{-1}KA^\tau P\|_\infty<1.$$

Condition (51) serves as its special case when taking $P=I$. It is easy to observe that the criterion in Ref. [4] is identical with our condition (54) when $A=0$.

III. CONCLUDING REMARKS

In this paper, we have established some sufficient criteria for delay-independent global asymptotic or exponential stability of delayed neural networks. Our results generalize some existing results in the literature, as stated in the above remarks. Our main results are formulated in Theorems 1–5 and their corollaries. Among these results, conditions (8), (12), (30), (32), and (54) are five basic ones that are independent of one another. To illustrate their independence, in the following we will give some simple examples, where only one condition is satisfied and the other four fail. For simplicity, in what follows, we consider a DNN with only two neurons, and further assume that all $p_i = 1$, i.e., $P = I$ and $C = K = I$, unless otherwise stated.

(1) A two-neuron DNN with weight matrices

$$A = \begin{pmatrix} 0.55 & 0.1 \\ 0.2 & -0.2 \end{pmatrix}, \quad A^{\tau} = \begin{pmatrix} 0.33 & 0.16 \\ 0.1 & -0.15 \end{pmatrix},$$

only satisfies condition (8) for both Euclidean and spectral norms, the other four conditions are not satisfied.

(2) For matrices

$$A = \begin{pmatrix} 0.3 & 0.15 \\ 0.05 & -0.1 \end{pmatrix}, \quad A^{\tau} = \begin{pmatrix} 0.1 & 0.5 \\ 0.6 & 0.2 \end{pmatrix},$$

only condition (12) holds.

(3) The choice of matrices

$$A = \begin{pmatrix} 0.5 & 0.2 \\ 0.1 & 0.3 \end{pmatrix}, \quad A^{\tau} = \begin{pmatrix} 0.5 & 0.7 \\ 1.0 & 0.5 \end{pmatrix},$$

$$K = \begin{pmatrix} 0.4 & \\ & 0.25 \end{pmatrix}, \quad C = \begin{pmatrix} 0.74 & \\ & 0.55 \end{pmatrix},$$

only satisfy condition (30) for the case of $r_1 = r_2 = 1/2$.

(4) If we select weight matrices as

$$A = \begin{pmatrix} -0.5 & 0.05 \\ 0.1 & -0.1 \end{pmatrix}, \quad A^{\tau} = \begin{pmatrix} 1.0 & 0.9 \\ 0.2 & 0.05 \end{pmatrix},$$

then only condition (32) is satisfied.

(5) If we select weight matrices as

$$A = \begin{pmatrix} 0.1 & 0.25 \\ 0.15 & 0.4 \end{pmatrix}, \quad A^{\tau} = \begin{pmatrix} 0.2 & 0.4 \\ 0.1 & 0.3 \end{pmatrix},$$

then only condition (54) is satisfied.

ACKNOWLEDGMENTS

This work was supported by the Natural Science Foundation of China under Grant No. 69901002, and supported in part by BASICS, Center of Basic Studies in Computing Science, sponsored by Shanghai Education Commission. BASICS is affiliated with the Department of Computer Science of Shanghai Jiao Tong University.

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